Remarks on bilinear estimates in the Sobolev spaces (Harmonic Analysis and Nonlinear Partial Differential Equations)

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By

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Abstract

The boundedness of integral operators of convolution type in the weighted Lebesgue spaces and the Leibniz rule of order one in one dimensional case are studied. As a byproduct, a simple proof of the fact that the standard Sobolev space $H^s(\mathbb{R}^n)$ forms an algebra for $s > n/2$ is given. Moreover, a simple proof of the Leibniz rule is also given, where a remarkable cancellation property is observed in the standard Littlewood-Paley argument.

§ 1. Introduction

We study the boundedness of integral operators of convolution type in the weighted Lebesgue space and the Leibniz rule of order one in one dimensional case.

To illustrate the first problem, we revisit the standard property that the Sobolev space $H^s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$ forms an algebra for $s > n/2$ from the point of view from the weighted $L^2(\mathbb{R}^n)$-boundedness of convolution. The corresponding bilinear estimate in the Sobolev space takes the form

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}$$

with $s > n/2$, where

$$\|u\|_{H^s} = \|(1-\Delta)^{s/2}u\|_{L^2} = \|(1+|\xi|^2)^{s/2}\hat{u}\|_{L^2},$$

$$\hat{u}(\xi) = \mathfrak{F}u(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp(-ix\cdot\xi)u(x)dx,$$

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and Δ is the Laplacian in $\mathbb{R}^n$. The bilinear estimate of this type was may be traced back at least to the paper by Saut and Temam [11]. More precisely, if $a, b, c \in \mathbb{R}$ satisfy either

$$a+b+c \geq n/2, \quad a+b>0, \quad b+c>0, \quad c+a>0$$

or

$$a+b+c > n/2, \quad a+b \geq 0, \quad b+c \geq 0, \quad c+a \geq 0,$$

then the following bilinear estimate holds

$$\|uv\|_{H^{-a}} \lesssim \|u\|_{H^b} \|v\|_{H^c}. \tag{1.1}$$

There are many papers on further refinements and improvements on this subject as well as various applications to nonlinear partial differential equations. (see for instance [3, 4, 6, 7, 8, 9, 10, 11, 12, 13] and references therein.)

One of the purpose in this paper is to give a simple and elementary proof of (1.1), which avoids paradifferential technique for instance.

By a duality argument, (1.1) is equivalent to the trilinear estimate of the form

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{\omega(\xi+\eta)^a \omega(\xi)^b \omega(\eta)^c} \hat{u}(\xi) \hat{v}(\eta) \hat{w}(\xi+\eta)\, d\eta\, d\xi \right| \\ &\leq C \|\hat{u}\|_{L^2} \|\hat{v}\|_{L^2} \|\hat{w}\|_{L^2}. \end{aligned} \tag{1.2}$$

with $\omega(\xi) = (1+|\xi|^2)^{1/2}$. More generally, we obtain the following estimate.

Theorem 1.1. *Let $2 \leq p \leq \infty$ and let w_0, w_1, w_2 be nonnegative, continuous functions on $[0, \infty)$ satisfying*

$$w_1(r) \geq C w_1(R), \quad w_2(r) \geq C w_2(R)$$

for all r and R with $0 \leq r \leq R$, and

$$M_1 \equiv \sup_{r>0} w_0(r) \|w_1(|\cdot|) w_2(|\cdot|)\|_{L^p(B(r))} < \infty, \tag{1.3}$$

$$M_2 \equiv \sup_{r>0} w_1(r) w_2(r) \|w_0(|\cdot|)\|_{L^p(B(r))} < \infty., \tag{1.4}$$

where $B(r) = \{x \in \mathbb{R}^n; |x| \leq r\}$. Then, the trilinear estimate

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w_0(|x+y|)\ w_1(|x|)\ w_2(|y|)\ |f(x+y) g(x) h(y)|\, dx\, dy \\ &\leq C(M_1+M_2) \|f\|_{L^{p'}} (\|g\|_{L^{p'}} \|h\|_{L^p} + \|g\|_{L^p} \|h\|_{L^{p'}}) \end{aligned} \tag{1.5}$$

holds for all $f \in L^{p'}(\mathbb{R}^n)$, g, $h \in L^p(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$, where p' is the dual exponent defined by $1/p + 1/p' = 1$.

The second purpose of this paper is to prove the fractional Leibniz rule of order one:

$$\|D(fg) - (Df)g - f(Dg)\|_{L^p(\mathbb{R})} \lesssim \|D^\theta f\|_{L^{p_1}(\mathbb{R})}\|D^{1-\theta} g\|_{L^{p_2}(\mathbb{R})}, \tag{1.6}$$

where $D^\alpha f = \mathfrak{F}^{-1}[|\cdot|^\alpha \hat{f}]$ for $\alpha \in \mathbb{R}$ and $1 < p, p_1, p_2 < \infty$ which satisfy $1/p = 1/p_1 + 1/p_2$ and $0 < \theta < 1$. The fractional Leibniz rule of order less than one:

$$\|D^\alpha(fg) - (D^\alpha f)g - f(D^\alpha g)\|_{L^p(\mathbb{R})} \lesssim \|D^{\alpha\theta} f\|_{L^{p_1}(\mathbb{R})}\|D^{\alpha(1-\theta)} g\|_{L^{p_2}(\mathbb{R})}$$

with $0 < \alpha < 1$ is proved by Kenig, Ponce, and Vega in [5]. Their proof is based on the decomposition of $D^\alpha(fg) - (D^\alpha f)g - f(D^\alpha g)$ to the tractable High-Low-High interaction part, the same frequency interaction part, and the remainder. In the case of (1.6), we can cancel the remainder part completely by a decomposition to positive frequency part and negative frequency part. Then (1.6) is obtained relatively easily.

Theorem 1.2. *For $1 < p, p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $0 < \theta < 1$,*

$$\|D(fg) - f(Dg) - (Df)g\|_{L^p(\mathbb{R})} \lesssim \|f\|_{\dot{H}^\theta_{p_1}(\mathbb{R})}\|g\|_{\dot{H}^{1-\theta}_{p_2}(\mathbb{R})}$$

where $\dot{H}^\theta_p = D^{-\theta} L^p$.

§ 2. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 by an estimate of [2].

Lemma 2.1 ([2]). *Let $2 \le p < \infty$ and let $\widetilde{w}_0$, $\widetilde{w}_1$, $\widetilde{w}_2$ be nonnegative, continuous functions on $[0, \infty)$ satisfying*

$$M_3 \equiv \sup_{r>0} \widetilde{w}_0^\#(2r)\widetilde{w}_2(r)\|\widetilde{w}_1(|\cdot|)\|_{L^p(B(r))} < \infty,$$
$$M_4 \equiv \sup_{r>0} \widetilde{w}_0^\#(2r)\widetilde{w}_1(r)\|\widetilde{w}_2(|\cdot|)\|_{L^p(B(r))} < \infty,$$

where

$$\widetilde{w}_0^\#(r) = \sup_{0\le\rho\le r} \widetilde{w}_0(\rho).$$

Then, the trilinear estimate

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widetilde{w}_0(|x+y|)\ \widetilde{w}_1(|x|)\ \widetilde{w}_2(|y|)\ |f(x+y)g(x)h(y)|\,dx\,dy$$
$$\le (M_3 + M_4)\|f\|_{L^p}\|g\|_{L^{p'}}\|h\|_{L^{p'}}$$

holds for all $f \in L^p(\mathbb{R}^n)$, g, $h \in L^{p'}(\mathbb{R}^n)$.

Proof of Theorem 1.1. By the Hölder and Minkowski inequalities, we obtain

$$
\begin{aligned}
&\iint_{|x|\le|y|} w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy\\
&\le C\iint_{|x|\le|y|} w_0(|x+y|)w_1(|x|)w_2(|x|)|f(x+y)g(x)h(y)|\,dx\,dy\\
&= C\iint w_0(|y|)w_1(|x|)w_2(|x|)|f(y)g(-x)h(x+y)|\,dx\,dy.
\end{aligned}
$$

Let $\widetilde{w}_0(\cdot)=1$, $\widetilde{w}_1(\cdot)=w_1(\cdot)w_2(\cdot)$, and $\widetilde{w}_2(\cdot)=w_0(\cdot)$. Then $\widetilde{w}_0$, $\widetilde{w}_1$, and $\widetilde{w}_2$ satisfy the condition of Lemma 2.1 with $M_3=M_1$ and $M_4=M_2$. Then by Lemma 2.1,

$$
\begin{aligned}
&\iint w_0(|y|)w_1(|x|)w_2(|x|)|f(y)g(-x)h(x+y)|\,dx\,dy\\
&\le C(M_1+M_2)\|f\|_{L^{p'}}\|g\|_{L^{p'}}\|h\|_{L^p}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\iint_{|x|\ge|y|} w_0(|x+y|)w_1(|x|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy\\
&\le C\iint_{|x|\ge|y|} w_0(|x+y|)w_1(|y|)w_2(|y|)|f(x+y)g(x)h(y)|\,dx\,dy\\
&\le C(M_1+M_2)\|f\|_{L^{p'}}\|g\|_{L^p}\|h\|_{L^{p'}}.
\end{aligned}
$$

Summing those inequalities, we have (1.5). □

Remark. The condition of Theorem 1.1 is more natural than the condition of Lemma 2.1, since two of w_0, w_1, and w_2 must not be almost increasing unbounded functions for the boundedness of (1.2). Indeed, if w_0 and w_1 almost increasing unbounded function;

$$
w_0(r)\le Cw_0(R),\quad w_1(r)\le Cw_1(R),\quad 0\le r\le R
$$

then

$$
\begin{aligned}
&\int w_0(|x+y|)w_1(|x|)\chi_{B(1)}(x+y+N)\chi_{B(1)}(x+N)dx\\
&\ge C^{-2}w_0(N-1)w_2(N-1)\int\chi_{B(1)}(x+y)\chi_{B(1)}(x)dx\\
&\to\infty\quad\text{as}\quad N\to\infty
\end{aligned}
$$

for $|y|\le 1$. This shows that (1.5) fails with $\chi_{B(1)}(\cdot+N)$, $\chi_{B(1)}(\cdot+N)$, and $\chi_{B(1)}(\cdot)$ as f, g, and h.

Remark. Theorem 1.1 gives a sharp sufficient condition of $a, b, c \in \mathbb{R}$ for the bilinear estimate for Sobolev norms (1.1); When $b, c \geq 0$, in the corresponding integral inequality,

$$M_1 = \sup_{r>0}(1+r)^{-a}\|(1+|\cdot|)^{-b-c}\|_{L^2(B(r))},$$
$$M_2 = \sup_{r>0}(1+r)^{-b-c}\|(1+|\cdot|)^{-a}\|_{L^2(B(r))}.$$

If a, $b+c \neq n/2$,

$$M_1 \lesssim \sup_{r\geq 0}(1+r)^{-a-\max(b+c-n/2,0)}, \tag{2.1}$$

$$M_2 \lesssim \sup_{r\geq 0}(1+r)^{-b-c-\max(a-n/2,0)}. \tag{2.2}$$

If $b+c = n/2$ and $a > 0$,

$$M_1 \lesssim \sup_{r\geq 0}(1+r)^{-a}\sqrt{\log(1+r)} < \infty, \tag{2.3}$$

If $a = n/2$ and $b+c > 0$,

$$M_2 \lesssim \sup_{r\geq 0}(1+r)^{-b-c}\sqrt{\log(1+r)} < \infty. \tag{2.4}$$

Then (2.1) - (2.4) show the optimality of the sufficient condition of (1.1), since we may exchange a, b, and c by a duality argument.

§ 3. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2 with a fundamental Littlewood-Paley argument.

§ 3.1. Preliminary

In this section, we use the following notation. Let $f : \mathbb{R} \to \mathbb{C}$ and $k \in \mathbb{Z}$. Let $\mathcal{S}$ is a set of rapidly decreasing functions. We define $\varphi \in \mathcal{S}$ with $0 \leq \varphi \leq 1$, supp $\varphi(x) \subset (-2, -2^{-1}) \cup (2^{-1}, 2)$, and $\sum_{k\in\mathbb{Z}} \varphi_k(x) = 1$ if $x \neq 0$, where $\varphi_k(x) = \varphi(2^{-k}x)$. Let $\psi \in \mathcal{S}$ satisfy $0 \leq \psi \leq 1$, $\psi = 1$ on $B(1)$, and $\psi = 0$ on $B(2)^c$. We also define

$$P_+ f = \mathfrak{F}^{-1}[\chi_{\mathbb{R}_{\geq 0}}\mathfrak{F}f], \quad P_- f = \mathfrak{F}^{-1}[\chi_{\mathbb{R}_{\leq 0}}\mathfrak{F}f], \quad P_k f = \mathfrak{F}^{-1}[\varphi_k \mathfrak{F}f]$$

and $P_{+,n} = P_+ P_n$, $P_{-,n} = P_- P_n$. Let

$$M[f](x) = \sup_{r>0}\frac{1}{2r}\int_{B(r)} |f(x-y)|dy.$$

Moreover, $\partial = d/dx$, $D^s = \mathfrak{F}^{-1}|\xi|^s\mathfrak{F}[\cdot]$ for any $s \in \mathbb{R}$. For any $s \in \mathbb{R}$ and $1 \le p \le \infty$, let $\dot{H}^s_p = D^{-s}L^p$ and

$$\|f\|_{\dot{H}^s_p} = \|D^s f\|_{L^p}.$$

It is well known that for $f \in \dot{H}^s_p$,

$$\|f\|_{\dot{H}^s_p} \sim \|f\|_{\dot{F}^s_{p,2}} = \|2^{ks}P_k f\|_{L^p(l^2_k)}.$$

Here, we collect some basic estimates.

Lemma 3.1 ([14]). *Let $R > 0$ and $f \in \mathcal{S}$ such that* $\operatorname{supp}\hat{f} \subset B(R)$. *Then for any $x \in \mathbb{R}$,*

$$R^{-1}\sup_{y\in\mathbb{R}}\langle Ry\rangle^{-1}\Big|f'(x-y)\Big| \lesssim \sup_{y\in\mathbb{R}}\langle Ry\rangle^{-1}|f(x-y)| \lesssim Mf(x).$$

Lemma 3.2 ([14]). *Let $R > 0$ and $f \in \mathcal{S}$ such that* $\operatorname{supp}\hat{f} \subset B(R) \subset \mathbb{R}$. *Then for any $x \in \mathbb{R}$ and $s \in \mathbb{R}$,*

$$|D^s f(x)| \lesssim R^s Mf(x).$$

Proof. Though this lemma seems to be elementary, we give a proof for definiteness. By Lemma 3.1, the inequality is obtained as follows

$$\begin{aligned}
|D^s f(x)| &= |(D^s\mathfrak{F}^{-1}[\psi(R^{-1}\cdot)] * f(x)| \\
&\lesssim \|\langle R\cdot\rangle\mathfrak{F}^{-1}[|R^{-1}\cdot|^s\psi(R^{-1}\cdot)]\|_{L^1}R^sMf(x) \\
&= \|\langle R\cdot\rangle\mathfrak{F}^{-1}[|\cdot|^s\psi](R\cdot)R\|_{L^1}R^sMf(x) \\
&\lesssim \|\langle\cdot\rangle D^s\mathfrak{F}^{-1}\psi\|_{L^1}R^sMf(x).
\end{aligned}$$

□

Lemma 3.3 ([1]). *Let $1 < p < \infty$ and $1 < q \le \infty$. Then M is a bounded operator on $L^p(l^q)$.*

Lemma 3.4. *Let $1 < p < \infty$. Then P_+ and P_- are bounded operators on $L^p(\mathbb{R})$.*

Proof. The proof is a slight modification of the proof of the equivalence of $\dot{H}^0_p$ and $\dot{F}^0_{p,2}$. For instance, see [15]. □

§ 3.2. Proof of Theorem 1.2

We calculate

$$\begin{aligned}
D(fg) - (Df)g - f(Dg) &= (D - i\partial)(fg) - ((D - i\partial)f)g - f((D - i\partial)g) \\
&= -2i\big(P_+\partial(fg) - (P_+\partial f)g - f(P_+\partial g)\big) \\
&= -2i\big(P_+[(\partial f)g + f(\partial g)] - (P_+\partial f)g - f(P_+\partial g)\big).
\end{aligned}$$

Moreover

$$\begin{aligned} P_+(\partial f)g - (P_+\partial f)g &= P_+((P_+ + P_-)(\partial f)g) - (P_+ + P_-)((P_+\partial f)g) \\ &= P_+((P_-\partial f)g) - P_-((P_+\partial f)g) \\ &= P_+((P_-\partial f)P_+g) - P_-((P_+\partial f)P_-g). \end{aligned}$$

With this representation, the High-High-High interaction part has been cancelled out. Then it is enough to show

$$\|P_+((P_-\partial f)P_+g)\|_{F^0_{p,2}(\mathbb{R})} \lesssim \|D^\theta f\|_{L^{p_1}(\mathbb{R})}\|D^{1-\theta}g\|_{L^{p_2}(\mathbb{R})}.$$

We divide $P_+((P_-\partial f)P_+g)$ into the High-High-Low interaction part and the High-Low-High interaction part as follows:

$$\begin{aligned} P_+((P_-\partial f)P_+g) &= P_+\sum_{k\in\mathbb{Z}}\sum_{|j-k|\le 2}(P_{-,j}\partial f)(P_{+,k}g) + P_+\sum_{k\in\mathbb{Z}}\sum_{j\le k-3}(P_{-,j}\partial f)(P_{+,k}g) \\ &= P_+(S_1 + S_2). \end{aligned}$$

We remark that the High-Low-High interaction part

$$P_+\sum_{k\in\mathbb{Z}}\sum_{j\ge k+3}P_{-,j}\partial f P_{+,k}g$$

has been canceled out.

S_1 Estimate: High-High-Low Interaction
By Lemmas 3.2 and 3.4 and Hölder inequality

$$\begin{aligned} \Big\|\sum_{k\in\mathbb{Z}}(P_{-,k}\partial f)(P_{+,k}g)\Big\|_{L^p(\mathbb{R})} &\le \Big\|\sum_{k\in\mathbb{Z}}2^k M[P_{-,k}f]|P_{+,k}g|\Big\|_{L^p(\mathbb{R})} \\ &\le \|MP_-f\|_{\dot F^\theta_{p_1,2}(\mathbb{R})}\|P_+g\|_{\dot F^{1-\theta}_{p_2,2}(\mathbb{R})} \\ &\lesssim \|f\|_{\dot H^\theta_{p_1}(\mathbb{R})}\|g\|_{\dot H^{1-\theta}_{p_2}(\mathbb{R})}. \end{aligned}$$

This shows $\|S_1\|_{L^p(\mathbb{R})} \lesssim \|f\|_{\dot H^\theta_{p_1}(\mathbb{R})}\|g\|_{\dot H^{1-\theta}_{p_2}}$.

S_2 Estimate: High-Low-High Interaction
We compute

$$P_{+,k}S_2 = P_{+,k}S_{2,k} = P_{+,k}\sum_{j=k-2}^{k+2}\sum_{l=-\infty}^{j-3}(\partial P_{-,l}f)(P_{+,j}g).$$

Since $\operatorname{supp} \hat{S}_{2,k} \subset B(2^{k+3})$, by the same calculation of the proof of Lemma 3.2, $|P_{+,k}S_2| \lesssim M[S_{2,k}]$ pointwise. Then by Lemmas 3.3 and 3.4 and the equivalence $L^p \sim \dot{F}^0_{p,2}$,

$$\begin{aligned}
\|P_+S_2\|_{L^p} &\sim \|P_{+,k}S_2\|_{L^p(l_k^2)} \\
&\lesssim \|M[S_{2,k}]\|_{L^p(l_k^2)} \\
&\lesssim \left\| \sum_{j=-\infty}^{k-3} |\partial P_{-,j}f| \; |P_{+,k}g| \right\|_{L^p(l_k^2)} \\
&\lesssim \left\| \sum_{j=-\infty}^{k-3} 2^j M[P_{-,j}f] \; |P_{+,k}g| \right\|_{L^p(l_k^2)} \\
&\lesssim \left\| 2^{(1-\theta)k} |P_{+,k}g| \sum_{j=-\infty}^{k-3} 2^{(1-\theta)(j-k)} 2^{\theta j} M[P_{-,j}f] \right\|_{L^p(l_k^2)} \\
&\lesssim \left\| 2^{(1-\theta)k} |P_{+,k}g| \; \left\| 2^{\theta j} M[P_{-,j}f] \right\|_{l_j^\infty} \right\|_{L^p(l_k^2)} \\
&\lesssim \|P_-f\|_{\dot{F}^\theta_{p_1,\infty}} \|P_+g\|_{\dot{F}^{1-\theta}_{p_2,2}} \\
&\lesssim \|f\|_{\dot{H}^\theta_{p_1}} \|g\|_{\dot{H}^{1-\theta}_{p_2}}.
\end{aligned}$$

This shows the desired estimate.

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